Understanding the Law of Propagation of Uncertainties

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The Law of Propagation of Uncertainties

$$u_c^2(y) = \boxed{\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)} + \boxed{2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)}$$

- This is from the ISO and BIPM Guide to the Expression of Uncertainty in Measurement (GUM)
- It applies to measurement models of the form:

$$Y = f(X_1, X_2, X_3, \ldots, X_i)$$

Things you might already "know"

$$u_c^2(y) = \boxed{\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)} + \boxed{2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)}$$

Has a sensitivity coefficient
Adding in quadrature (% or units)

Averages reduce by $1/\sqrt{n}$

This term is to do with correlation

At the end of this talk, you should understand…

- The Law of Propagation of Uncertainties
  - Why the sensitivity coefficient is central to the law
  - How to determine the sensitivity coefficients
  - What 'adding in quadrature' really means
- How many readings you should average
  - What the uncertainty associated with an average is
  - How you know you have enough readings
  - When you've taken too many readings
- What correlation is
  - What the uncertainties are for averaging partially correlated data
  - How to estimate covariance from numerical and experimental data
  - How to deal with not knowing what the correlation is

The Law of Propagation of Uncertainties

Ignore this part for now…

$$u_c^2(y) = \sum_{i=1}^{n}\left(\boxed{\frac{\partial f}{\partial x_i}}\right)^2 u^2(x_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)$$

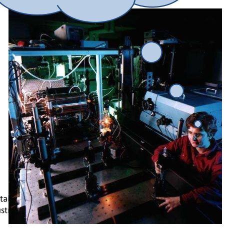
Combined Uncertainty

Sensitivity Coefficient

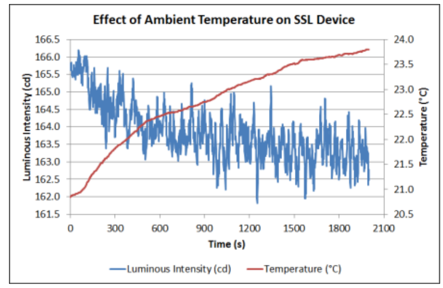
Individual Component's Uncertainty

Do an experiment to find out!

How sensitive is my result to this?

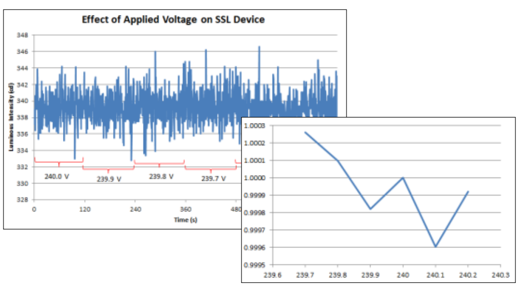


Determining sensitivity from experiment



- Sensitivity ~ -0.5% / °C

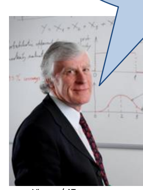
Determining sensitivity from experiment

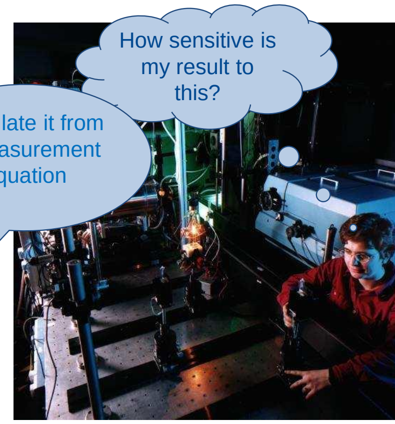


- Effect is negligible

How sensitive is my result to this?

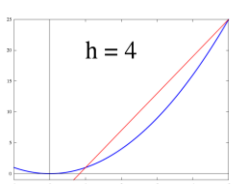
Calculate it from a measurement equation





$$y = f\left(x_1, x_2, x_3, \ldots\right)$$

$$c_{x_i} = \frac{\partial f}{\partial x_i}$$



The local slope:
Translating a change in one parameter to a change in the other

Example

$$V_S = V_{light} - V_{dark}$$

$$\frac{\partial V_S}{\partial V_{light}} = 1 \qquad \frac{\partial V_S}{\partial V_{dark}} = -1$$

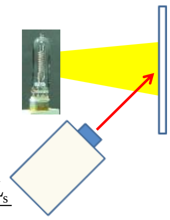
Very simple case

Example 2

Luminance meter calibration

$$L_\mathrm{s} = \frac{E_\mathrm{FEL}\beta_{0-45}}{\pi}\frac{d_\mathrm{cal}^2}{d_\mathrm{use}^2}$$

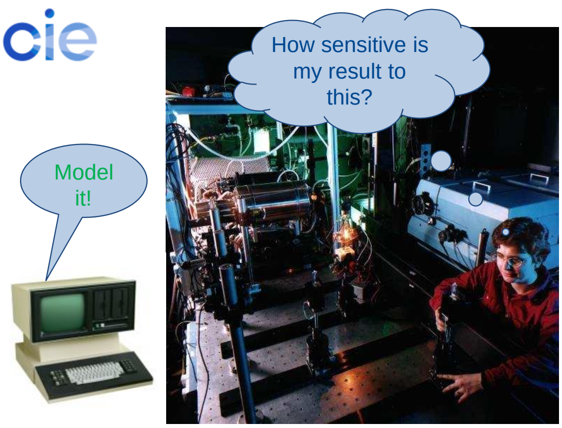
$$\frac{\partial L_\mathrm{s}}{\partial d_\mathrm{use}} = -2\frac{E_\mathrm{FEL}\beta_{0-45}}{\pi}\frac{d_\mathrm{cal}^2}{d_\mathrm{use}^3} = \frac{-2L_\mathrm{s}}{d_\mathrm{use}}$$



Inverse square law leads to a 2

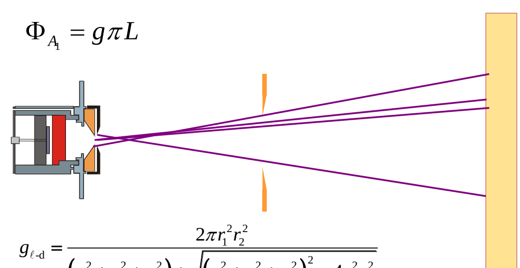
When measurement equations don't work well:

- Because you can't write a relationship
- Because it's too difficult to differentiate
- Because it's a program not an equation



Example: Radiance through double aperture system

$$\Phi_{A_1} = g\pi L$$



$$g_{\ell\text{-d}} = \frac{2\pi r_1^2 r_2^2}{\left(r_1^2+r_2^2+x^2\right)+\sqrt{\left(r_1^2+r_2^2+x^2\right)^2-4r_1^2r_2^2}}$$

Can be achieved by differentiating …

$$g_{\ell\text{-d}} = \frac{2\pi r_1^2 r_2^2}{\left(r_1^2+r_2^2+x^2\right)+\sqrt{\left(r_1^2+r_2^2+x^2\right)^2-4r_1^2r_2^2}} \qquad g = 2\pi r_1^2 r_2^2 \alpha^{-1}$$

$$\beta = \sqrt{\left(r_1^2+r_2^2+d^2\right)^2-4r_1^2r_2^2}$$

$$\alpha = \left(r_1^2+r_2^2+d^2\right)+\beta$$

$$\frac{\partial g}{\partial r_1} = \frac{4\pi r_1 r_2^2}{\alpha}\left[1-\frac{r_1^2}{\alpha}\left(\frac{\alpha-2r_2^2}{\beta}\right)\right]$$

$$\frac{\partial g}{\partial r_2} = \frac{4\pi r_1^2 r_2}{\alpha}\left[1-\frac{r_2^2}{\alpha}\left(\frac{\alpha-2r_1^2}{\beta}\right)\right]$$

$$\frac{\partial g}{\partial d} = \frac{-4\pi r_1^2 r_2^2 d}{\alpha\beta}$$

Or, by modelling…

	A	B	C	D	E	F	G
4				Uncertainty in diameter or distance			
5	**This first section calculates g for the straightforward case**			microns	metres		3.1E-10
6	radius of first aperture / m	0.0025	5	1	1E-06		57.986%
7	radius of second aperture / m	0.0015	3	1	1E-06		
8	distance between apertures / m	0.3	300	40	0.00004		
9							
10	sum of squares	0.0900085			**Uncertainty associated with g**		
11	4 r1^2 r2^2	5.625E-11			due to first aperture		0.0400%
12	square root	0.0900085	beta		due to second aperture		0.0667%
13	bottom line	0.180017	alpha		due to distance		-0.0267%
14	g	4.908E-10			***Combined uncertainty associated with g***		**0.082%**
15							
16	**This section recalculates g, changing one parameter at a time**						
17		Changing first aperture	Changing second aperture	Changing distance			
18	radius of first aperture / m	0.0025005	0.0025	0.0025			
19	radius of second aperture / m	0.0015	0.0015005	0.0015			
20	distance between apertures / m	0.3	0.3	0.30004			
21							
22	sum of squares	0.090008503	0.090008502	0.090032502			
23	4 r1^2 r2^2	5.62725E-11	5.62875E-11	5.625E-11			
24	square root	0.090008502	0.090008501	0.090032501			
25	bottom line	0.180017005	0.180017003	0.180065003			
26	g	4.910E-10	4.912E-10	4.907E-10			
27		0.0400%	0.0667%	-0.0267%			
28							
29							

Another example…

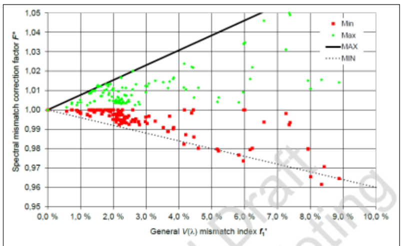
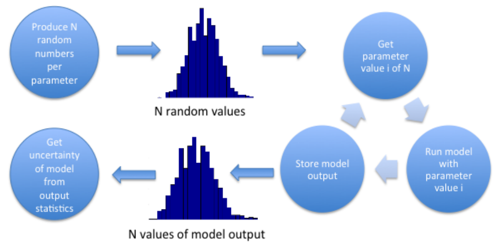


Figure C.4 – Spectral mismatch correction factors (SMCF) for phosphor-type white LEDs and different f_1' values of photometers

From CIE DIS 025/E:2014 – get your copy very soon!

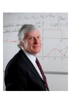
Or, using Monte Carlo simulation



Sensitivity Coefficients

Experimentally
Vary in the lab and see what happens

Numerically
Vary in the model and see what happens

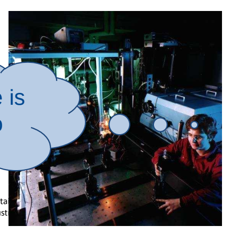


Mathematically

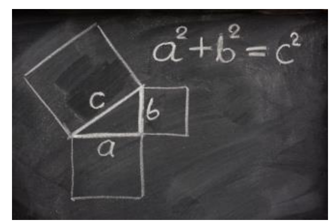
$$c_{x_i} = \frac{\partial f}{\partial x_i}$$

All are acceptable

How sensitive is my result to this?



Adding in Quadrature



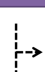
$$u_c^2(y) = \boxed{\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)} + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)$$

Why quadrature?

Uncertainty

Describes the spread

Error



Difference to the (unknowable) true value

It's very unlikely that all will be maximum in same direction!

Combining distributions leads to the quadrature rule

Another way of putting it…

- Variance is the square of the standard uncertainty.
- It is a principle of statistics that it is variances that are added when combining effects.

Uncertainty expression

Relative uncertainty:
5 mW m^{-2} nm^{-1} $\pm$ 0.2 %
i.e. uncertainty expressed as a percentage

Absolute uncertainty:
5 mW m^{-2} nm^{-1} $\pm$ 0.01 mW m^{-2} nm^{-1}
i.e. uncertainty expressed in the native measurement units

Adding in quadrature for absolute models

$$y = A + B + C \qquad \frac{\partial y}{\partial A} = \frac{\partial y}{\partial B} = \frac{\partial y}{\partial C} = 1$$

$$u^2(y) = u^2(A) + u^2(B) + u^2(C)$$

Adding in quadrature for absolute models

$$y = A - B$$

$$\frac{\partial y}{\partial A} = 1 \qquad \frac{\partial y}{\partial B} = -1$$

$$u^2(y) = u^2(A) + (-1)^2 u^2(B)$$

$$u^2(y) = u^2(A) + u^2(B)$$

What behaves like an absolute model?

Distance = Measured distance + aperture thickness

Signal = Light signal – Dark signal

These are of the form y = A + B and y = A – B and can be dealt with absolutely: Y, A and B all have the same units.

Uncertainty expression

Relative uncertainty:
5 mW m^{-2} nm^{-1} $\pm$ 0.2 %
i.e. uncertainty expressed as a percentage

Absolute uncertainty:
5 mW m^{-2} nm^{-1} $\pm$ 0.01 mW m^{-2} nm^{-1}
i.e. uncertainty expressed in the native measurement units

Adding in quadrature for relative models

$$y = A \times B \qquad \frac{\partial y}{\partial A} = B = \frac{y}{A} \qquad \frac{\partial y}{\partial B} = A = \frac{y}{B}$$

$$u^2(y) = \left(\frac{y}{A}\right)^2 u^2(A) + \left(\frac{y}{B}\right)^2 u^2(B)$$

$$\left(\frac{u(y)}{y}\right)^2 = \left(\frac{u(A)}{A}\right)^2 + \left(\frac{u(B)}{B}\right)^2$$

Adding in quadrature for relative models

$$y = A/B \qquad \frac{\partial y}{\partial A} = \frac{1}{B} = \frac{y}{A} \qquad \frac{\partial y}{\partial B} = \frac{-A}{B^2} = \frac{-y}{B}$$

$$u^2(y) = \left(\frac{y}{A}\right)^2 u^2(A) + \left(\frac{-y}{B}\right)^2 u^2(B)$$

$$\left(\frac{u(y)}{y}\right)^2 = \left(\frac{u(A)}{A}\right)^2 + \left(\frac{u(B)}{B}\right)^2$$

Adding in quadrature for relative models

$$y = A/B^2 \qquad \frac{\partial y}{\partial A} = \frac{1}{B^2} = \frac{y}{A} \qquad \frac{\partial y}{\partial B} = \frac{-2A}{B^3} = \frac{-2y}{B}$$

$$u^2(y) = \left(\frac{y}{A}\right)^2 u^2(A) + \left(\frac{-2y}{B}\right)^2 u^2(B)$$

$$\left(\frac{u(y)}{y}\right)^2 = \left(\frac{u(A)}{A}\right)^2 + \left(\frac{2u(B)}{B}\right)^2$$

What behaves like a relative model?

- Most radiometric equations!
- Anything where uncertainties are in %

Example

$$\frac{\partial L_s}{\partial E_{\mathrm{FEL}}} = \frac{L_s}{E_{\mathrm{FEL}}}$$

$$\frac{\partial L_s}{\partial \beta_{0°:45°}} = \frac{L_s}{\beta_{0°:45°}}$$

$$L_s = \frac{E_{\mathrm{FEL}}\beta_{0°:45°}}{\pi}\frac{d_{\mathrm{cal}}^2}{d_{\mathrm{use}}^2} \qquad \frac{\partial L_s}{\partial d_{\mathrm{use}}} = \frac{-2L_s}{d_{\mathrm{use}}}$$

$$\left(\frac{u(L_s)}{L_s}\right)^2 = \left(\frac{u(E_{\mathrm{FEL}})}{E_{\mathrm{FEL}}}\right)^2 + \left(\frac{u(\beta_{0°:45°})}{\beta_{0°:45°}}\right)^2 + \left(\frac{2u(d_{\mathrm{cal}})}{d_{\mathrm{cal}}}\right)^2$$

At the end of this talk, you should understand...

- The Law of Propagation of Uncertainties
  - Why the sensitivity coefficient is central to the law
  - How to determine the sensitivity coefficients
  - What 'adding in quadrature' really means
- How many readings you should average
  - What the uncertainty associated with an average is
  - How you know you have enough readings
  - When you've taken too many readings
- What correlation is
  - What the uncertainties are for averaging partially correlated data
  - How to estimate covariance from numerical and experimental data
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Applying the Law of Propagation of Uncertainties to an average

Averaging three readings

$$E_{\mathrm{M}} = \frac{E_1 + E_2 + E_3}{3} \qquad \frac{\partial E_{\mathrm{M}}}{\partial E_1} = \frac{\partial E_{\mathrm{M}}}{\partial E_2} = \frac{\partial E_{\mathrm{M}}}{\partial E_3} = \frac{1}{3}$$

$$u^2(E_{\mathrm{M}}) = \left(\frac{1}{3}\right)^2 u^2(E_1) + \left(\frac{1}{3}\right)^2 u^2(E_2) + \left(\frac{1}{3}\right)^2 u^2(E_3)$$

Assumption: $u(E_1) = u(E_2) = u(E_3) = u(E_i)$

$$u^2(E_{\mathrm{M}}) = 3\left(\frac{1}{3}\right)^2 u^2(E_i)$$

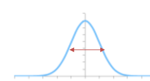
$$u^2(E_{\mathrm{M}}) = \left(\frac{u(E_i)}{\sqrt{3}}\right)^2 \qquad u(E_{\mathrm{M}}) = \left(\frac{u(E_i)}{\sqrt{n}}\right)$$

What is the uncertainty associated with each reading?

We made the assumption:

$u(E_1) = u(E_2) = u(E_3) = u(E_i)$

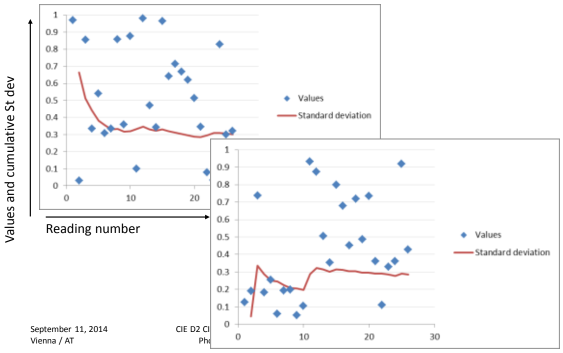
Uncertainty



The uncertainty in each measurement is a draw from a probability distribution

We need to know the Standard Deviation

The dangers of taking a standard deviation of a small number of readings



What you should do

Use more readings!

Take more measurements during a test

$$u(E_M) = \left(\frac{u(E_i)}{\sqrt{n}}\right)$$

Estimate of uncertainty

More to average

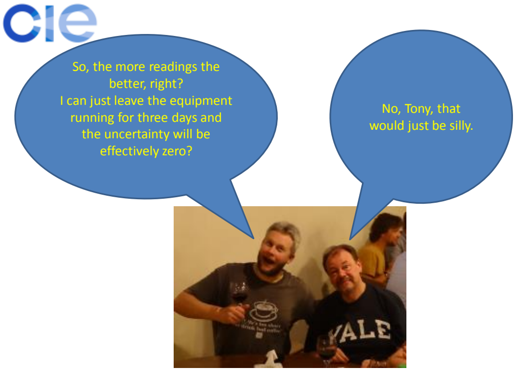
When there are few readings…

Increase the Estimate

- We need to increase in standard uncertainty when there are fewer measurements
- The revised GUM will have the following:

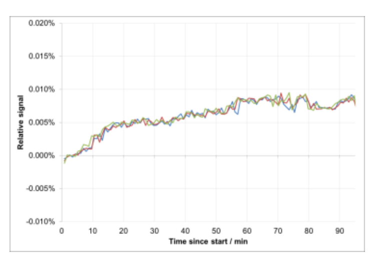
$$u^2_{\text{light,mean}} = \boxed{\frac{N-1}{N-3}}\left(\frac{s_{\text{light}}}{\sqrt{N}}\right)^2$$

- If N = 5, the standard deviation is multiplied by ~1.41
- If N = 25, the standard deviation is multiplied by ~1.04



- Use more readings…
- But only when it is worth it…
- It is important to know when to stop!

Influence of drift…



How can we tell the influence of drift???

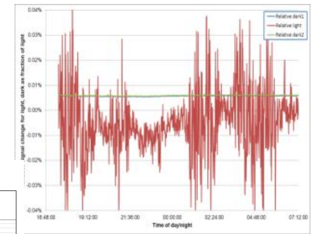
The Allan Variance

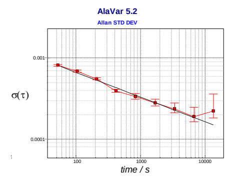
- Take a series of measurements at regular time intervals Δt
- The Allan Variance calculates the difference between successive data points, squares them and takes the average, before dividing the answer by two.

$$\sigma^2(\Delta t)=\frac{1}{2}\left[\frac{1}{N/2}\left[(y_2-y_1)^2+(y_4-y_3)^2+\ldots+(y_N-y_{N-1})^2\right]\right]$$

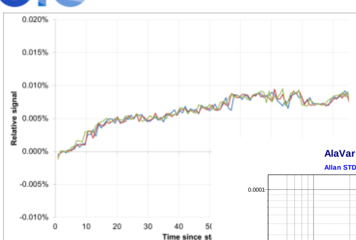
- The pairs of data are then averaged, so now you have half as many data points with a time interval of $2\Delta t$ and the Allan Variance is calculated again.
- This is repeated, each time averaging the pairs that were used before.
- Plot the Allan Variance vs Time on a log-log scale graph.
- For white noise, the Allan deviation lies on a straight line on this log-log plot, with a slope of -0.5.

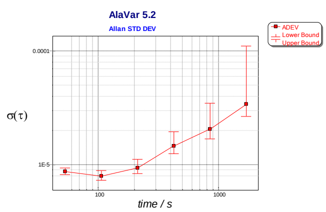
The uncertainties don't keep dropping forever





The Allan Deviation – software





www.alamath.com

[Note: third-party software, not official CIE recommendation]

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The Law of Propagation of Uncertainties

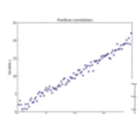
$$u_c^2(y)=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)+\boxed{2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)}$$

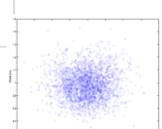
Correlation occurs when there is something in common between the different quantities, or between the measured values being combined.

"Covariance" is a measure of how much two quantities vary together, i.e. the "something in common" bit.

Correlation: Type A and Type B methods

Type A: From the data





Type B: From knowledge

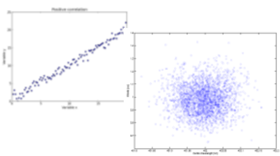
$$E_i = E_{\text{True}} + S + R_i$$

This is where the correlation comes from!

Systematic Effects!

Correlation: Type A and Type B methods

Type A: From the data



Correlation coefficient:

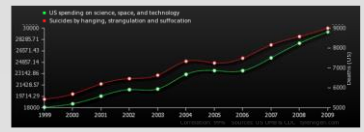
$$r(x,y)=\frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{s_X}\right)\left(\frac{Y_i-\bar{Y}}{s_Y}\right)$$

$$u(x,y)=u(x)u(y)r(x,y)$$

$$u_c^2(y)=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)+\boxed{2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)}$$

Correlation example

US spending on science, space, and technology
correlates with
Suicides by hanging, strangulation and suffocation



	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
US spending on science, space, and technology Millions of todays dollars (US OMB)	18,079	18,594	19,753	20,734	20,831	23,029	23,597	23,584	25,525	27,731	29,449
Suicides by hanging, strangulation and suffocation Deaths (US) (CDC)	5,427	5,688	6,198	6,462	6,635	7,336	7,248	7,491	8,161	8,578	9,000

Correlation: 0.992082

Permalink - Mark as interesting (5,147) - Not interesting (2,370)

Conclusion: beware of spurious correlations!

Correlation: Type A and Type B methods

Type B: From knowledge

$$E_i=E_{\text{True}}+S+R_i$$

Calculate covariance

This is where the correlation comes from!
Systematic Effects!

Averaging partially correlated data

$$E_M=\frac{E_1+E_2+E_3}{3}$$

$$E_i=E_{\text{True}}+S+R_i$$

$$E_M=\frac{3E_{\text{True}}}{3}+\frac{3S}{3}+\frac{R_1+R_2+R_3}{3}$$

$$u^2(E_M)=u^2(S)+\left(\frac{u(R_i)}{\sqrt{3}}\right)^2$$

$$u^2(E_M)=u^2(S)+\left(\frac{u(R_i)}{\sqrt{n}}\right)^2$$

Systematic and random effects: Lamp measured multiple times

Systematic effects	Random effects
Reference calibration	Noise
Alignment	Lamp current fluctuation
Lamp current setting	
Temperature sensitivities	

$$S,u(S) \qquad R_i,u(R_i)$$

Covariance Matrix

Start with the law of propagation of uncertainty:

$$u_c^2(y)=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i)+2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}u(x_i,x_j)$$

We can write this in matrix form as:

$$u^2(y)=\boldsymbol{C}_y\boldsymbol{U}_x\boldsymbol{C}_y^{\top}$$

Where the C_y are column vectors of Sensitivity coefficients:

$$\boldsymbol{C}_y=\left(\frac{\partial f}{\partial x_1}\quad\frac{\partial f}{\partial x_2}\quad\cdots\quad\frac{\partial f}{\partial x_n}\right)$$

And U_x is the covariance matrix, which takes the form:

$$\boldsymbol{U}_E=\begin{array}{c c}
 & \begin{array}{cccc}1 & 2 & \cdots & n\end{array}\\
\begin{array}{c}1\\2\\\vdots\\n\end{array} & \begin{bmatrix}u^2(E_1) & u(E_1,E_2) & \cdots & u(E_1,E_n)\\ u(E_2,E_1) & u^2(E_2) & \cdots & u(E_2,E_n)\\ \vdots & \vdots & \ddots & \vdots\\ u(E_n,E_1) & u(E_n,E_2) & \cdots & u^2(E_n)\end{bmatrix}
\end{array}$$

Covariance matrix with absolute variance (squared uncertainty) and covariance

Say this covariance matrix relates to the measurement of a spectrum:

- Diagonals represent the variance associated with spectral irradiance value at λ_i.
- Off-diagonal terms represent the covariance associated with the measured values at two different wavelengths

Covariance Matrix

If there are both multiplicative and additive effects, a model may look like:

$$E(\lambda_i) = E_{\mathrm{T}}(\lambda_i)(1+S)(1+R_i) + \tilde{s} + \tilde{r}_i$$

Where:

- S could be distance and alignment effects;
- R_i could be lamp or current stability;
- $\tilde{s}$ could be a common dark reading subtracted from all wavelengths;
- $\tilde{r}_i$ could be dark signal noise.

Then, for the ith row and jth column in the matrix:

$$\tilde{U}_{E,ij} = \begin{cases} E_i^2\left[u^2(S)+u^2(R_i)\right]+u^2(\tilde{s})+u^2(\tilde{r}_i) & (i=j) \\ E_i E_j u^2(S)+u^2(\tilde{s}) & (i \neq j). \end{cases}$$

Earlier, when we weren't considering correlation, the sign of the sensitivity coefficient was irrelevant.

But when we consider correlations, the sign of the sensitivity coefficient is important!

Negative sensitivity coefficients

Eg: Signal = Light - Dark

$$y = A - B \qquad \frac{\partial y}{\partial A} = 1 \quad \frac{\partial y}{\partial B} = -1$$

$$u^2(y) = u^2(A) + u^2(B) + 2(1)(-1)u(A,B)$$

$$u^2(y) = u^2(A) + u^2(B) - 2u(A,B)$$

Correlation reduces uncertainty

Positive sensitivity coefficients

Eg: Distance = measured distance + aperture thickness

$$y = A + B \qquad \frac{\partial y}{\partial A} = 1 \quad \frac{\partial y}{\partial B} = 1$$

$$u^2(y) = u^2(A) + u^2(B) + 2(1)(1)u(A,B)$$

$$u^2(y) = u^2(A) + u^2(B) + 2u(A,B)$$

Correlation increases uncertainty

Don't know what the covariance is?

- Think of the worst-case scenario

Does correlation increase or decrease the uncertainty?

Treat as "all random" and "all systematic"

Summary and conclusion

Sensitivity Coefficients

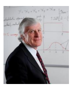
$$u_c^2(y) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2 u^2(x_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} u(x_i, x_j)$$

Experimentally

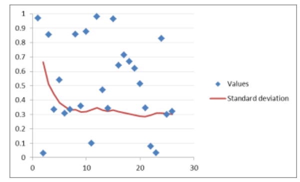
Mathematically

Numerically

How sensitive is my result to this?



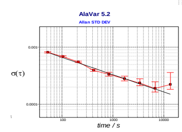
Taking enough readings, but not too many

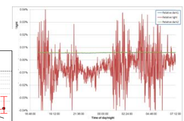


Better estimate of uncertainty

More to average

$$u(E_M) = \left(\frac{u(E_i)}{\sqrt{n}}\right)$$





Allan Deviation

Correlation

Correlation occurs when there is something in common between the different parameters, or between the measurements being combined.

$$E_i = E_{\text{True}} + S + R_i$$

This is where the correlation comes from!

Systematic Effects!

Thank you for your kind attention

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And…. thanks Emma!